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Particle emission in the de Sitter universe for massless fields with spin

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Abstract. The thermal flux emitted by a de Sitter universe due to interaction with massless fields with spin is calculated by examining the field equations of these fields. The field equations are obtained by perturbing the metric with the various fields and obtaining linearised equations. The expression for the power radiated indicates the absence of spin-half particle emission.

1. Introduction

The suggestion by Hawking (1974) that black holes and other event horizons such as those in the de Sitter universe (Gibbons and Hawking 1977) may emit thermal radiation has been an exciting development in recent years. In an earlier work (Lohiya and Panchapakesan 1978) we examined the thermal flux of scalar particles in a de Sitter universe. In this paper we derive expressions for the thermal flux for fields with spin. Besides electromagnetic (spin-1) we also consider the gravitational (spin-2) and the neutrino (spin-$\frac{1}{2}$) fields. This involves obtaining and solving linearised field equations for various spin fields in the de Sitter metric.

In obtaining the field equations we use the formalism of Newman and Penrose (1962), and following the method of Teukolsky (1973) we obtain linearized field equations. The partial differential equation is also separable in this case and the resulting radial equation is used to obtain the absorption coefficient for waves incident on the horizon.

In § 2 we describe the de Sitter metric in the Newman–Penrose formalism and obtain the field equations for spin-2, 1 and $\frac{1}{2}$ in the same formalism by using the method of perturbations developed by Teukolsky (1973) in the case of black holes.

In § 3 we discuss the separation of variables of the field equation and obtain the equation for the radial coordinate. The equations for all the spins can be combined and written in a generalised radial equation.

In § 4 we calculate the absorption coefficient from the radial equation. We follow the method used by Page (1976) and determine the solution valid far away from the horizon and compare its ingoing and outgoing parts. We comment briefly on the results in § 5.

2. Field equations

The conventional form of the de Sitter metric is

$$ds^2 = [1-(r^2/a^2)]\,dt^2 - [1-(r^2/a^2)]^{-1}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.1}$$

where a is related to the cosmological constant Λ and scalar curvature R by $R = 4\Lambda = 12/a^2$.

We find it convenient to perform the calculations in the v, r, θ, ϕ coordinates. where

$$v = t + r^* = \tfrac{1}{2}a\ \ln[(a+r)/(a-r)] + t. \tag{2.2}$$

We have

$$ds^2 = [1-(r^2/a^2)]\,dv^2 - 2\,dv\,dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.3}$$

A special choice for the null tetrads which simplify the spin coefficients in these coordinates is

$$\begin{aligned} l^\mu &= [0, 1, 0, 0] \\ n^\mu &= [-1, -\tfrac{1}{2}[1-(r^2/a^2)], 0, 0] \\ m^\mu &= \frac{1}{\sqrt{2}}\left[0, 0, \frac{1}{r}, \frac{i}{r\sin\theta}\right]. \end{aligned} \tag{2.4}$$

Defining the Newman–Penrose (NP) coefficients as

$$\begin{aligned} \alpha &= \tfrac{1}{2}(l_{\mu;\nu}n^\mu \bar{m}^\nu - m_{\mu;\nu}\bar{m}^\mu \bar{m}^\nu) \\ \beta &= \tfrac{1}{2}(l_{\mu;\nu}n^\mu m^\nu - m_{\mu;\nu}\bar{m}^\mu m^\nu) \\ \gamma &= \tfrac{1}{2}(l_{\mu;\nu}n^\mu n^\nu - m_{\mu;\nu}\bar{m}^\mu n^\nu) \\ \epsilon &= \tfrac{1}{2}(l_{\mu;\nu}n^\mu l^\nu - m_{\mu;\nu}\bar{m}^\mu l^\nu) \\ \kappa &= l_{\mu;\nu}m^\mu l^\nu \qquad\qquad \pi = -n_{\mu;\nu}\bar{m}^\mu l^\nu \\ \lambda &= -n_{\mu;\nu}\bar{m}^\mu \bar{m}^\nu \qquad \rho = l_{\mu;\nu}m^\mu \bar{m}^\nu \\ \mu &= -n_{\mu;\nu}\bar{m}^\mu m^\nu \qquad \sigma = l_{\mu,\nu}m^\mu m^\nu \\ \nu &= -n_{\mu;\nu}\bar{m}^\mu n^\nu \qquad \tau = l_{\mu;\nu}m^\mu n^\nu \end{aligned}$$

it is not difficult to show that the only non-vanishing spin coefficients for the unperturbed metric are

$$\begin{aligned} \beta &= -\alpha = \cot\theta/2\sqrt{2}r \qquad \gamma = -r/2a^2 \\ \mu &= -[1-(r^2/a^2)]/2r \qquad \rho = -1/r. \end{aligned} \tag{2.5}$$

Although the Ricci tensor in the de Sitter space–time is non-zero, all its tetrad components can be easily seen to vanish by using the defining properties of the tetrad, namely $ln = -\bar{m}m = 1$; other products are zero and $R_{\mu\nu} = \Lambda g_{\mu\nu}$. (This is seen to imply that the Goldberg–Sachs theorem holds for empty space–times with a non-vanishing cosmological constant along the lines given by Newman and Penrose (1962).)

Using the NP equations with equations (2.4) and (2.5) the unperturbed de Sitter space–time turns out to be a type-0 space–time—with all the Weyl-Tensor tetrad components vanishing. The de Sitter–Schwarzchild space–time, given by

$$ds^2 = [1-(2m/r)-(r^2/a^2)]\,dt^2 - [1-(2m/r)-(r^2/a^2)]^{-1}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

is, however, a type-D space–time with the non-vanishing component of the Weyl-tensor projection being $\psi_2 = -m/r^3$. It is therefore convenient to consider the de Sitter space–time as the limit of a type-D space–time as m approaches zero. The analysis of Teukolsky (1973) can be taken over for our case as indicated below.

We perturb the geometry by expressing all the NP quantities as a sum of the unperturbed and the perturbed parts, e.g.

$$D = D^0 + D^{\mathrm{p}} \qquad \psi_0 = \Psi_0^0 + \Psi_0^{\mathrm{p}}, \text{ etc.}$$

The unperturbed quantities satisfy

$$\psi_0^0 = \psi_1^0 = \psi_2^0 = \psi_3^0 = \psi_4^0 = 0 \qquad \epsilon^0 = \kappa^0 = \lambda^0 = \nu^0 = \pi^0 = \sigma^0 = \tau^0 = 0. \tag{2.6}$$

Following Teukolsky (1973) we then have

$$[(D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})(\Delta - 4\gamma + \mu) - (\delta + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)(\bar{\delta} + \bar{\pi} - 4\alpha) - 3\Psi_2]\psi_0^{\mathrm{p}} \tag{2.7}$$
$$= \text{source terms}$$

and

$$[(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \rho) - (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\Psi_2]\psi_4^{\mathrm{p}}$$
$$= \text{source term.}$$

Dropping the superscript p in our case, the equations reduce for the source-free case to

$$[(D - 5\rho)(\Delta - 4\gamma + \mu) - (\delta + 2\alpha)(\bar{\delta} - 4\alpha)]\Psi_0 = 0 \tag{2.8}$$

and

$$[(\Delta + 2\gamma + 5\mu)(D - \rho) - (\bar{\delta} + 2\alpha)(\delta - 4\alpha)]\psi_4 = 0$$

with

$$D \equiv \partial/\partial r \qquad \Delta \equiv (-\partial/\partial v) - \tfrac{1}{2}[1 - (r^2/a^2)]\,\partial/\partial r \tag{2.9}$$

and

$$\delta \equiv \frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right).$$

The equation (2.8) is the field equation in NP form for a spin-2 field in the linearized approximation.

In the case of electromagnetic and neutrino field perturbations we neglect the change in the background geometry and the equations of Teukolsky (1973) reduce in our case to

$$[(D - 3\rho)(\Delta + \mu - 2\gamma) - \delta(\bar{\delta} - 2\alpha)]\Phi_0 = 0$$

and (2.10)

$$[(\Delta + 3\mu)(D - \rho) - \bar{\delta}(\delta + 2\beta)]\Phi_2 = 0$$

for the electromagnetic field.

In the case of a neutrino or spin-$\frac{1}{2}$ massless field, we get

$$[(D - 2\rho)(\Delta - \gamma + \mu) - (\delta - \alpha)(\bar{\delta} - \alpha)]\chi_0 = 0$$

and (2.11)

$$[(\Delta - \gamma + 2\mu)(D - \rho) - (\bar{\delta} - \alpha)(\delta - \alpha)]\chi_1 = 0$$

along with equation (2.9) in all cases.

3. Separation of variables, the general radial equation and the boundary conditions

The field equations (2.8), (2.10) and (2.11) separate out just as in the case of the Kerr black hole. We use the spin-weight spherical harmonics ${}_pY_l^m$ of Goldberg *et al* (1967). The quantity p takes the values $\pm s$ corresponding to different helicity states of the spin-s field where s can take values 0, $\frac{1}{2}$, 1 and 2. The ${}_pY_l^m$ satisfy the equation

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)+\left((l-p)(l+p+1)+p-p^2\cot^2\theta-\frac{2mp\cos\theta}{\sin^2\theta}-\frac{m^2}{\sin^2\theta}\right)\right]{}_pY_l^m=0 \tag{3.1}$$

where m is the azimuthal quantum number and $l \geqslant s$.

We find that all the massless fields satisfy a general equation in the variables v, r:

$$\left\{\frac{\partial^2}{\partial r\,\partial v}+\frac{(s+p+1)}{r}\frac{\partial}{\partial v}+\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)\frac{\partial^2}{\partial r^2}-\frac{(l-s)(l+s+1)}{2r^2}\right.$$
$$\left.-\left[\frac{(p+1)r}{a^2}-\frac{(s+1)}{r}\left(1-\frac{r^2}{a^2}\right)\right]\frac{\partial}{\partial r}-\frac{(s+p+1)(s+p+2)}{a^2}\right\}\Psi=0 \tag{3.2}$$

where ${}_pY_l^m\psi(r, v)$ is the general solution.

We wish to have the radial equation in t, r, θ, ϕ coordinates corresponding to a detector sensitive to positive frequencies with respect to time t. We therefore write

$$\begin{aligned}\psi(v, r) &= \exp(-\mathrm{i}\omega t)R(r)\\ &= \exp[-\mathrm{i}\omega(v-r^*)]R(r)\end{aligned} \tag{3.3}$$

and find for the radial equation (using $z \equiv r/a$)

$$\begin{aligned}z^2(1-z^2)^2R_{,zz}&-[2(p+1)z^3-2(s+1)(1-z^2)z](1-z^2)R_{,z}\\ &-\{(1-z^2)[(l-s)(l+s+1)+(s+p+1)(s+p+2)z^2]\\ &-(\omega az)^2+2\mathrm{i}a\omega zp\}R=0.\end{aligned} \tag{3.4}$$

The conformally coupled scalar field has been discussed earlier (Lohiya and Panchapakesan 1978), where the nature of the boundary condition to be imposed has also been discussed. We recall briefly the following conditions.

We define (Gibbons and Hawking 1977)

$$r=a(1+UV)(1-UV)^{-1} \qquad \exp(2t/a)=-V/U$$

which correspond to

$$V=\pm e^{u/a} \qquad \text{and} \qquad U=\mp e^{-v/a}$$

with $u=t-r^*$ and $v=t+r^*$.

The Gibbons–Hawking vacuum is defined such that we have $e^{-\mathrm{i}\omega V}$ at the future horizon which is a function of $u=t-r^*$ only and corresponds to outgoing boundary conditions. On the past horizon we have functions which are of positive frequency with respect to U and this is known to give the Hawking radiation (Unruh 1976, Lohiya and Panchapakesan 1978).

Accordingly, we impose the outgoing boundary condition at the future horizon.

4. Absorption probability and thermal flux

The exact solutions of the radial equation (3.4) seem difficult to find. We can, however, determine the absorption probability at low frequency by using a solution obtained by matching the solution valid at the origin and the solution satisfying the outgoing boundary condition at the future horizon (Page 1976). To this effect, we note that near the origin, $z \ll 1$, equation (3.4) reduces to

$$R_{,zz}+\frac{2(s+1)}{z}R_{,z}+\left((\omega a)^2-\frac{2\mathrm{i}\omega ap}{z}-\frac{(l-s)(l+s+1)}{z^2}\right)R=0. \tag{4.1}$$

This has a general solution of the form (Morse and Feshbach 1953)

$$\begin{aligned}R=&A\,\mathrm{e}^{kz}z^{l-s}{}_1F_1(l+1+p,2l+2;-2kz)\\&+B\,\mathrm{e}^{kz}z^{-l-s-1}{}_1F_1(-l+p,-2l;-2kz)\end{aligned} \tag{4.2}$$

where $k=\mathrm{i}a\omega$. The constants A and B are evaluated by matching the solution satisfying outgoing boundary conditions at the horizon.

To discuss the solution near the horizon ($z\simeq 1$) we define a z^* coordinate by $\mathrm{d}z^*=\mathrm{d}z/(1-z^2)$. We also change to the function $f(l,r)$ defined by $R(l,r)=f(l,r)/z$. The equation (3.4) reduces, near $z=1$, to

$$f_{,z^*z^*}+f_{,z^*}(-2p)-(k^2+2pk)f=0 \tag{4.3}$$

which gives two independent solutions

$$f_1\xrightarrow{z\to1}\mathrm{e}^{-kz^*}\qquad\text{and}\qquad f_2\xrightarrow{z\to1}\mathrm{e}^{(2p+k)z^*}. \tag{4.4}$$

f_1 and f_2 correspond to incoming and outgoing boundary conditions at the horizon respectively.

To get the solutions with the desired behaviour at $z=1$ we change to $x=1/z$ and define $g(x)=(x^2-1)^{-c/2}f(x)$. The factor $(x^2-1)^{-c/2}$ will behave like e^{cz^*} for $z\simeq1$. The function $g(x)$ satisfies, near the origin, the equation

$$(1-x^2)g_{,xx}-2(\mu+1)xg_{,x}+(l-\mu)(l+\mu+1)g=0 \tag{4.5}$$

where $\mu\equiv c-s$.

The two independent solutions to this equation are (Bateman 1953)

$$f_1\sim\frac{\Gamma(1+l+\mu)}{\Gamma(l+\frac{3}{2})}z^{l+\mu+1}(z^{-2}-1)^{c_1/2}{}_2F_1(1+\tfrac12(\mu+l),\tfrac12(\mu+l+1),l+\tfrac32;z^2) \tag{4.6}$$

with $c_1=k$ and

$$\begin{aligned}f_2\sim&\,2^{-l-1}\Gamma(-\tfrac12-l)z^{l+1-\mu}(z^{-2}-1)^{c_2/2}{}_2F_1(1+\tfrac12(l-\mu),1-\tfrac12(l-\mu);\tfrac12-l;z^2)+\dots\\&+2^l\frac{\Gamma(\frac12+l)}{(z^{-2}-1)^{\mu}}z^{-l-\mu}\frac{(z^{-2}-1)^{c_2/2}}{\Gamma(1+l-\mu)}{}_2F_1(-\tfrac12(l-\mu),\tfrac12(1-l-\mu);\tfrac12-l;z^2).\end{aligned} \tag{4.7}$$

c_1 and c_2 are fixed by the behaviour at the horizon and by the known exact solutions for the scalar case (Lohiya and Panchapakesan 1978).

We match the outgoing solution equation (4.7) with equation (4.2) as $z \to 0$ to determine the constants A and B and find

$$A = 2^{-l-1}\frac{\Gamma(-\frac{1}{2}-l)}{\Gamma(-l-\mu)} \qquad B = 2^{l}\frac{\Gamma(\frac{1}{2}+l)}{\Gamma(1+l-\mu)}. \tag{4.8}$$

It is convenient to drop one of the coefficients s or p by defining a new function (Teukolsky and Press 1974) given by

$$\begin{aligned} f_2^{\text{new}}(p=s) &= f_2^{\text{old}}(p=s) \\ f_2^{\text{new}}(p=-s) &= z^{2s} f_2^{\text{old}}(p=-s)(\equiv \rho^{-2s} f_2^{\text{old}}). \end{aligned} \tag{4.9}$$

Then

$$\begin{aligned} R = f/z + A\, \mathrm{e}^{kz} z^{l-p}\, {}_1F_1(l+1+p, 2l+2; -2kz) \\ + B\, \mathrm{e}^{kz} z^{-l-p-1}\, {}_1F_1(-l+p, -2l; -2kz). \end{aligned} \tag{4.10}$$

The asymptotic form of the confluent hypergeometric function enables us to put the solutions in the form (for $a\omega \gg r\omega \gg 1$)

$$R \sim Y_{\text{in}}\, \mathrm{e}^{-kz}/z^{1+2p} + Y_{\text{out}}\, \mathrm{e}^{kz}/z$$

where

$$\begin{aligned} Y_{\text{in}} &= \frac{(-k)^{-l-1-p} 2^{-p} \Gamma(l+1)}{\Gamma(-l-\mu)\Gamma(l+1+p)} - \frac{\Gamma(-l) 2^{-p} (-k)^{l-p}}{\Gamma(-l-p)\Gamma(1+l-\mu)} \\ Y_{\text{out}} &= \frac{(k)^{p-l-1} 2^{p} \Gamma(l+1)}{\Gamma(-l-\mu)\Gamma(l+1+p)} - \frac{k^{l+p}\Gamma(-p) 2^{p}}{\Gamma(-l+p)\Gamma(1+l-\mu)}. \end{aligned} \tag{4.11}$$

Following Teukolsky and Press (1974) the absorption probability Γ is given by

$$1 - \Gamma = |Y_{\text{in}} z_{\text{in}} / Y_{\text{out}} z_{\text{out}}| \tag{4.12}$$

where z_{in}, z_{out} are the same as Y_{in}, Y_{out} with p replaced by $-p$. Defining

$$\begin{aligned} A_1 &= \frac{\Gamma(-l)\Gamma(l+1-s)\Gamma(-l-k-3s)}{\Gamma(-l-s)\Gamma(l+1)\Gamma(l-k-3s+1)} \\ A_2 &= \frac{\Gamma(-l)\Gamma(l+1+s)\Gamma(-l-k-3s)}{\Gamma(-l+s)\Gamma(l+1)\Gamma(l-k-3s+1)} \\ B_1 &= \frac{\Gamma(-l)\Gamma(l+1+s)\Gamma(-k-l+s)}{\Gamma(-l+s)\Gamma(l+1)\Gamma(l+1-k+s)} \\ B_2 &= \frac{\Gamma(-l)\Gamma(l+1-s)\Gamma(-l-k+s)}{\Gamma(-l-s)\Gamma(l+1)\Gamma(l+1-k+s)} \end{aligned} \tag{4.13}$$

we find

$$1 - \Gamma = \left| \frac{[1 + \mathrm{i}(-1)^l (a\omega)^{2l+1} A_1][1 + \mathrm{i}(-1)^l (a\omega)^{2l+1} B_1]}{[1 - \mathrm{i}(-1)^l (a\omega)^{2l+1} A_2][1 - (-1)^l \mathrm{i}(a\omega)^{2l+1} B_2]} \right|. \tag{4.14}$$

The power spectra are given by the relation

$$a\frac{\mathrm{d}E}{\mathrm{d}\omega\, \mathrm{d}t} = \sum_{l,p} \frac{(2l+1)(a\omega)\Gamma(l,\omega)}{(\exp(2\pi a\omega) - 1)} \tag{4.15}$$

where $k_{\mathrm{B}}\Gamma = 1/2\pi a$ is the Hawking temperature.

5. Discussion

The thermal flux is given by equations (4.14) and (4.15). It can be verified that they reduce to the previously derived result (Lohiya and Panchapakesan 1978) for the scalar case.

For the integral spins ($s = 1, 2$) $\Gamma \to 0$ for the lowest l value, unlike the scalar case where for $l = 0$, Γ was finite. The expression for the emitted flux is finite and well behaved for the cases $s = 1$ and 2.

The case of half-integral spin $s = \frac{1}{2}$ is, however, quite different. As both l and s are half-integral $\Gamma(-l-s)$ diverges in equation (4.14) and makes $A_1 = A_2 = B_1 = B_2 = 0$. This leads to $\Gamma = 0$ and hence there is no thermal flux in the massless fields of spin-half. This is in contrast to the case of the black hole calculated by Page (1976). The physical significance of the absence of spin-half thermal radiation is not clear to us and needs further investigation.

As all open universes ultimately tend to the de Sitter universe, the features observed in this investigation may have importance in cosmology.

One of the referees has suggested that the use of the Gibbons–Hawking and Unruh boundary condition on the past horizon requires further study. We are looking into this suggestion.

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